Mathematical Finance
Dylan Possamaï

Assignment 3

We fix throughout a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we are given a filtration $\mathbb{F}$, unless otherwise stated.

A martingale inequality

Let $(M_n)_{n\in\mathbb{N}}$ be an $(\mathbb{F},\mathbb{P})$-martingale in discrete-time such that $M_0 = 0$ and for any $n \in \mathbb{N}$

$$|M_{n+1} - M_n| \leq a_{n+1},\ \mathbb{P}\text{–a.s.}$$

for some sequence $(a_n)_{n\in\mathbb{N}^\star}$ of non-negative numbers satisfying $A^2 := \sum_{n=1}^{+\infty} a_n^2 < +\infty$.

1) Prove that M is bounded in $\mathbb{L}^2(\mathbb{R},\mathcal{F},\mathbb{P})$. Deduce that $M_n \longrightarrow_{n\to+\infty} M_\infty$, $\mathbb{P}$–almost surely and in $\mathbb{L}^2(\mathbb{R},\mathcal{F},\mathbb{P})$, for some M_∞ in $\mathbb{L}^2(\mathbb{R},\mathcal{F},\mathbb{P})$.

2) Show that for any $c > 0$
$$\mathbb{P}\bigg[\sup_{n\in\mathbb{N}} M_n \geq c\bigg] \leq \exp\bigg(-\frac{c^2}{2A^2}\bigg).$$
Hint: try to apply Doob's maximal inequality to $(\mathrm{e}^{\lambda M_n})_{n\in\mathbb{N}}$*, for some* $\lambda > 0$*. You may use the inequality* $\cosh(x) \leq \mathrm{e}^{x^2/2}$*,* $x \in \mathbb{R}$*.*

Brownian motion and stopping times

Let B be a one-dimensional $(\mathbb{F},\mathbb{P})$–Brownian motion. For $x \in [-1,1]$, we define $B_t^x = x + B_t$, $t \geq 0$ a Brownian motion 'started at x'. Let $\tau^x := \inf\{t > 0 : |B_t^x| \geq 1\}$ be the first time that it exits the interval $[-1,1]$.

1) Let g be a continuous function on $[-1,1]$. Show that the function $u : [-1,1] \longrightarrow \mathbb{R}$ defined by
$$u(x) := \mathbb{E}^{\mathbb{P}}\bigg[\int_0^{\tau_x} g(B_s^x)\mathrm{d}s\bigg],$$
is well-defined and continuous.

   Hint: start by showing that τ_x *is* $\mathbb{P}$*-integrable by considering the martingale* $\big((B_t^x)^2 - t\big)_{t\geq 0}$.

2) Suppose that v is a bounded function on $[-1,1]$ such that $v(-1) = v(1) = 0$, and furthermore the process M^x defined by
$$M_t^x := v\big(B^x_{t\wedge\tau^x}\big) + \int_0^{t\wedge\tau^x} g(B_s^x)\mathrm{d}s,\ t \geq 0,$$
is an $(\mathbb{F},\mathbb{P})$–local martingale for each $x \in [-1,1]$.

   Prove that $u = v$.

3) Suppose that v is a bounded function on $[-1,1]$ such that $v(-1) = v(1) = 0$ and it satisfies the second-order differential equation
$$\frac{1}{2}v''(x) = -g(x). \tag{0.1}$$
Show that $v = u$.

4) Replacing g by the Dirac delta mass δ_y at some point $y \in \mathbb{R}$, formally compute the solution v_y to (0.1). The function $v_y(x) =: G(x, y)$ is called Green's function. Can you find a solution to (0.1) for more general g, in terms of G?

Some SDEs

Let σ be a continuous positive function on $\mathbb{R}$, satisfying the following linear growth condition for some $K > 0$

$$|\sigma(x)| \leq K(1 + |x|),\ x \in \mathbb{R}.$$

Suppose that we have been given a one-dimensional $(\mathbb{F}, \mathbb{P})$–Brownian motion B, and a family of processes $(X^x)_{x \in \mathbb{R}}$ such that, for each $x \in \mathbb{R}$, the following stochastic differential equation is satisfied

$$X_t^x = x + \int_0^t \sigma(X_s^x) \mathrm{d}B_s,\ t \geq 0.$$

1) Prove that for each time $T > 0$ and each $p \geq 1$, there is a constant c (depending only on T, K and p but not on x) such that

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{0 \leq t \leq T} |X_T^x|^p\right] \leq c(1 + |x|^p).$$

2) Construct a pair (X, B), where B is another one-dimensional $(\mathbb{F}, \mathbb{P})$–Brownian motion, such that the following stochastic differential equation is satisfied

$$X_t = \int_0^t \mathrm{sgn}(X_s) \mathrm{d}B_s,\ t \geq 0,$$

where $\mathrm{sgn}(x) := -\mathbf{1}_{\{x \leq 0\}} + \mathbf{1}_{\{x > 0\}}$.